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On $\mathfrak{F}\Phi^*$-hypercentral subgroups of finite groups

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ABSTRACT

Let G be a finite group. We write $R(G)$ to denote the largest soluble normal subgroup of G and put $\Phi^*(G) = \Phi(R(G))$. We say that a chief factor H/K of G is non-Frattini (non-solubly-Frattini) if $H/K \nleq \Phi(G/K)$ (if $H/K \nleq \Phi^*(G/K)$, respectively). A chief factor H/K of G is called $\mathfrak{F}$-central in G provided $(H/K) \rtimes (G/C_G(H/K)) \in \mathfrak{F}$. A normal subgroup N of G is said to be $\mathfrak{F}\Phi$-hypercentral ($\mathfrak{F}\Phi^*$-hypercentral) in G if either $N = 1$ or $N \neq 1$ and there exists a chief series $1 = N_0 < N_1 < \cdots < N_t = N$ $(*)$ of G below N such that every non-Frattini (non-solubly-Frattini, respectively) factor N_i/N_{i-1} of Series $(*)$ is $\mathfrak{F}$-central in G. In this paper we analyze some properties and applications of $\mathfrak{F}\Phi$-hypercentral and $\mathfrak{F}\Phi^*$-hypercentral subgroups.

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1. Introduction

Throughout this paper, all groups are finite, G denotes a finite group and p is a prime. We write $R(G)$ to denote the largest soluble normal subgroup of G and put $\Phi^*(G) = \Phi(R(G))$.

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Let $\mathfrak{F}$ be a class of groups. If $1 \in \mathfrak{F}$, then we write $G^{\mathfrak{F}}$ to denote the intersection of all normal subgroups N of G with $G/N \in \mathfrak{F}$. The class $\mathfrak{F}$ is said to be a *formation* if either $\mathfrak{F} = \varnothing$ or $1 \in \mathfrak{F}$ and every homomorphic image of $G/G^{\mathfrak{F}}$ belongs to $\mathfrak{F}$ for any group G. The formation $\mathfrak{F}$ is said to be: *saturated* if $G \in \mathfrak{F}$ whenever $G/\Phi(G) \in \mathfrak{F}$; *solubly saturated* if $G \in \mathfrak{F}$ whenever $G/\Phi^*(G) \in \mathfrak{F}$; (*normally*) *hereditary* if $H \in \mathfrak{F}$ whenever $G \in \mathfrak{F}$ and H is a (normal) subgroup of G. It is clear that any saturated formation is solubly saturated.

It is well-known that the class $\mathfrak{N}$ of all nilpotent groups and the class $\mathfrak{U}$ of all supersoluble groups are hereditary saturated formations. The class of all quasinilpotent groups is a normally hereditary formation (see [19, Chapter X, Lemma 13.3]). It is also known that this formation is solubly saturated but not saturated (see Shemetkov [29] or [3, p. 97]).

All formations met in mathematical practice are normally hereditary. Note also in passing that normally hereditary saturated or solubly saturated formations are the most useful in applications of the theory of formations.

If H/K is a chief factor of G and G belongs to a formation $\mathfrak{F}$, then the semidirect product $(H/K) \rtimes (G/C_G(H/K))$ also belongs to $\mathfrak{F}$ (see Barnes and Kegel [6] or Proposition 1.5 in [7, Chapter IV]). On the other hand, if $\mathfrak{F}$ is a solubly saturated formation and $(H/K) \rtimes (G/C_G(H/K)) \in \mathfrak{F}$ for each chief factor H/K of G, then $G \in \mathfrak{F}$ (Shemetkov [28]). This circumstance is a motivation for the following concept (P. Schmid): A chief factor H/K is called *$\mathfrak{F}$-central in G* provided $(H/K) \rtimes (G/C_G(H/K)) \in \mathfrak{F}$, otherwise it is called *$\mathfrak{F}$-eccentric in G*.

We say that a chief factor H/K of G is *non-Frattini* (*non-solubly-Frattini* [33]) if $H/K \nleq \Phi(G/K)$ (if $H/K \nleq \Phi^*(G/K)$, respectively).

A normal subgroup N of G is said to be *$\mathfrak{F}$-hypercentral* (*$\mathfrak{F}\Phi$-hypercentral* [33]) in G if either $N = 1$ or $N \neq 1$ and every (every non-Frattini, respectively) chief factor of G below N is $\mathfrak{F}$-central in G. The *$\mathfrak{F}$-hypercentre* $Z_{\mathfrak{F}}(G)$ of G is the product of all normal $\mathfrak{F}$-hypercentral subgroups of G (see [7, p. 389]), the *$\mathfrak{F}\Phi$-hypercentre* $Z_{\mathfrak{F}\Phi}(G)$ of G is the product of all normal $\mathfrak{F}\Phi$-hypercentral subgroups of G [33].

It is well-known that if H is a normal subgroup of G and $\mathcal{H}_1$ and $\mathcal{H}_2$ are chief series of G below H, then there exists a one-to-one correspondence between the chief factors of $\mathcal{H}_1$ and those of $\mathcal{H}_2$ such that corresponding factors are G-isomorphic and such that the non-Frattini chief factors of $\mathcal{H}_1$ correspond to the non-Frattini chief factors of $\mathcal{H}_2$ (see, for example, Lemma 2.2 in [33]). Thus every normal subgroup of G contained in an $\mathfrak{F}$-hypercentral ($\mathfrak{F}\Phi$-hypercentral) subgroup of G is $\mathfrak{F}$-hypercentral ($\mathfrak{F}\Phi$-hypercentral, respectively) in G itself, and the product of any two $\mathfrak{F}$-hypercentral ($\mathfrak{F}\Phi$-hypercentral) subgroups of G is $\mathfrak{F}$-hypercentral ($\mathfrak{F}\Phi$-hypercentral, respectively) in G. As the analogue of the result (for the non-solubly-Frattini chief factors) is not known now, dealing with such chief factors it is convenient to use the following concept.

Definition 1.1. A normal subgroup N of G is said to be *$\mathfrak{F}\Phi^*$-hypercentral* in G if either $N = 1$ or $N \neq 1$ and there exists a chief series

$$1 = N_0 < N_1 < \cdots < N_t = N$$

of G below N such that every non-solubly-Frattini factor N_i/N_{i-1} of this series is $\mathfrak{F}$-central in G.

In this paper we analyze some properties and applications of $\mathfrak{F}\Phi$-hypercentral and $\mathfrak{F}\Phi^*$-hypercentral subgroups. Note that our Theorems A, B and D are true for any non-empty formation $\mathfrak{F}$.

Theorem A. *Let $\mathfrak{F}$ be any formation, E a normal subgroup of G.*

(i) *If $F^*(E)$ is $\mathfrak{F}\Phi^*$-hypercentral in G, then E is also $\mathfrak{F}\Phi^*$-hypercentral in G.*
(ii) *If $F^*(E)$ is $\mathfrak{F}$-hypercentral in G, then E is also $\mathfrak{F}$-hypercentral in G.*
(iii) *If E is soluble and $F(E)$ is $\mathfrak{F}\Phi$-hypercentral in G, then E is also $\mathfrak{F}\Phi$-hypercentral in G.*

In this theorem $F^*(E)$ is the generalized Fitting subgroup of E, that is, the largest normal quasinilpotent subgroup of E (see [19, Chapter X]).

Corollary 1.2. *(See Asaad [1] or Li [21].) Let E be a normal subgroup of G. If every chief factor of G below $F^*(E)$ is cyclic, then every chief factor of G below E is also cyclic.*

Since $F^*(E) = F(E)$ whenever E is soluble [19, X, Corollary 13.7], we get from Theorem A(ii) the following:

Corollary 1.3. *Let $\mathfrak{F}$ be any formation, E a soluble normal subgroup of G. If $F(E)$ is $\mathfrak{F}$-hypercentral in G, then E is also $\mathfrak{F}$-hypercentral in G.*

Now let A be any simple non-abelian group and let $\mathfrak{F}$ be the class of all groups W such that every composition factor of W is not isomorphic to A. It is clear that $\mathfrak{F}$ is a solubly saturated formation and $F(A) = 1$ is $\mathfrak{F}$-hypercentral in A. Nevertheless, A is not $\mathfrak{F}\Phi$-hypercentral in A. This obvious example shows that in general, where E is not soluble, Theorem A(iii) and Corollary 1.3 are not true.

Among other corollaries of Theorem A, there are the following:

Corollary 1.4. *Let $\mathfrak{F}$ be any solubly saturated formation. If $F^*(G)$ is $\mathfrak{F}\Phi^*$-hypercentral in G, then $G \in \mathfrak{F}$.*

Corollary 1.5. *Let $\mathfrak{F}$ be any solubly saturated formation and E a normal subgroup of G such that $F^*(E)$ is $\mathfrak{F}\Phi^*$-hypercentral in G. If $G/E \in \mathfrak{F}$, then $G \in \mathfrak{F}$.*

Recall that the *$\mathfrak{F}\Phi^*$-hypercentre* $Z_{\mathfrak{F}\Phi^*}(G)$ of G is the product of all normal $\mathfrak{F}\Phi^*$-hypercentral subgroups of G [33].

The symbol $\pi(\mathfrak{F})$ denotes the set of all primes p such that p divides $|G|$ for some group $G \in \mathfrak{F}$.

If $\mathfrak{F} = (1)$ is the formation of all identity groups, then $Z_{\mathfrak{F}\Phi}(G) = \Phi(G)$ is the subgroup Frattini of G. For general case, we prove on the basis of Theorem A(ii) the following result.

Theorem B. *Let $\mathfrak{F} \neq (1)$ be a non-empty formation and $\pi = \pi(\mathfrak{F})$. Then*

$$Z_{\mathfrak{F}\Phi}(G)/\Phi(G) = Z_{\mathfrak{F}}\big(G/\Phi(G)\big),$$

and $Z_{\mathfrak{F}\Phi}(G) = A \times B$, where $A = O_\pi(Z_{\mathfrak{F}\Phi}(G))$, $B = O_{\pi'}(\Phi(G))$ and $A/A \cap \Phi(G) \leqslant Z_{\mathfrak{F}}(G/A \cap \Phi(G))$.

Recall that $\mathrm{Soc}(G)$ denotes the product of all minimal normal subgroups of G whenever $G \neq 1$, and $\mathrm{Soc}(1) = 1$.

Theorem C. *Let $\mathfrak{F}$ be a normally hereditary formation containing all nilpotent groups and E a normal subgroup of G. Let $D = E \cap Z_{\mathfrak{F}\Phi}(G)$.*

(i) *If $\mathfrak{F}$ is solubly saturated and $E/E \cap Z_{\mathfrak{F}\Phi^*}(G) \in \mathfrak{F}$, then $E \in \mathfrak{F}$. Hence if $E \leqslant Z_{\mathfrak{F}\Phi^*}(G)$, then $E \in \mathfrak{F}$; in particular, if $G = Z_{\mathfrak{F}\Phi^*}(G)$, then $G \in \mathfrak{F}$.*
(ii) *If $\mathfrak{F}$ is saturated and $E/D \in \mathfrak{F}$, then $E \in \mathfrak{F}$. Hence if $E \leqslant Z_{\mathfrak{F}\Phi}(G)$, then $E \in \mathfrak{F}$; in particular, if $G = Z_{\mathfrak{F}\Phi}(G)$, then $G \in \mathfrak{F}$.*
(iii) *If $\mathfrak{F}$ is saturated and $\mathrm{Soc}(E/D) \leqslant Z_{\mathfrak{F}\Phi}(E/D)$, then $E \in \mathfrak{F}$.*

Corollary 1.6. *Let $\mathfrak{F}$ be a normally hereditary saturated formation containing all nilpotent groups and E a normal subgroup of G. If $E/E \cap \Phi(G) \in \mathfrak{F}$, then $E \in \mathfrak{F}$.*

Another proof of Corollary 1.6 can be found in [12] or in [30].

Corollary 1.7. *(See Shemetkov [31].) Let $\mathfrak{F}$ be a normally hereditary solubly saturated formation containing all nilpotent groups and E a normal subgroup of G. If $E/E \cap \Phi^*(G) \in \mathfrak{F}$, then $E \in \mathfrak{F}$.*

Corollary 1.8. *If $\mathfrak{F}$ is a normally hereditary solubly saturated formation, then $Z_{\mathfrak{F}}(G) \in \mathfrak{F}$.*

From Corollary 1.8 we get the following well-known result.

Corollary 1.9. *(See [7, Chapter IV, Theorem 6.15].) If $\mathfrak{F}$ is a normally hereditary saturated formation, then $Z_{\mathfrak{F}}(G) \in \mathfrak{F}$.*

Let $\mathfrak{F}$ be a class of groups. Then we say following [16,17] that G is a *quasi-$\mathfrak{F}$-group* if for every $\mathfrak{F}$-eccentric chief factor H/K of G, every automorphism of H/K induced by an element of G is inner. The symbol $\mathfrak{F}^*$ denotes the class of all quasi-$\mathfrak{F}$-groups. Therefore $\mathfrak{N}^*$ is the class of all quasinilpotent groups G, that is, for every chief factor H/K of G, every automorphism of H/K induced by an element of G is inner; $\mathfrak{U}^*$ is the class of all quasisupersoluble groups G, that is, for every non-cyclic chief factor H/K of G, every automorphism of H/K induced by an element of G is inner. In [16] (see also [17]) it is proved that for any normally hereditary saturated formation $\mathfrak{F}$ containing all nilpotent groups the class $\mathfrak{F}^*$ is a normally hereditary solubly saturated formation.

Therefore from Theorem C(i) we get

Corollary 1.10. *Let $\mathfrak{F}$ be a normally hereditary saturated formation containing all nilpotent groups and E a normal subgroup of G. If $E/E \cap Z_{\mathfrak{F}^*\Phi^*}(G) \in \mathfrak{F}^*$, then $E \in \mathfrak{F}^*$.*

Corollary 1.11. *Let E be a normal subgroup of G. If $E/E \cap Z_{\mathfrak{N}^*\Phi^*}(G)$ is quasinilpotent, then E is also quasinilpotent.*

Corollary 1.12. *Let E be a normal subgroup of G. If $E/E \cap Z_{\mathfrak{U}^*\Phi^*}(G)$ is quasisupersoluble, then E is quasisupersoluble.*

A maximal subgroup M of G is said to be *$\mathfrak{F}$-abnormal* in G if $G/M_G \notin \mathfrak{F}$. Following [30], we use $\Delta^{\mathfrak{F}}(G)$ to denote the intersection of all $\mathfrak{F}$-abnormal maximal subgroups of G.

Theorem D. *For any formation $\mathfrak{F}$ we have*

$$\Delta^{\mathfrak{F}}(G) = Z_{\mathfrak{F}\Phi}(G).$$

From Theorems B, C(ii) and D we get

Corollary 1.13. *(See Ballester-Bolinches [2].) For any hereditary saturated formation $\mathfrak{F}$ the subgroup $(\Delta^{\mathfrak{F}}(G))^{\mathfrak{F}}$ is nilpotent.*

Corollary 1.14. *(See Feng and Chang [8].) Let $\mathfrak{F}$ be a saturated formation, $\Delta = \Delta^{\mathfrak{F}}(G)$ and $\pi = \pi(\mathfrak{F})$. Then*

$$\Delta^{\mathfrak{F}}\bigl(G/O_{\pi}(\Delta)\bigr) = \Phi\bigl(G/O_{\pi}(\Delta)\bigr) \leqslant O_{\pi'}\bigl(G/O_{\pi}(\Delta)\bigr).$$

Note that if E is a quasinilpotent normal subgroup of G and $E \cap \Phi(G) = 1$, then E is the direct product $E = E_1 \times \cdots \times E_t$ of some minimal normal subgroups $E_1, \ldots, E_t$ of G (see Lemma 2.15 below). Hence $C_G(E) = C_G(E_1) \cap \cdots \cap C_G(E_t)$. Therefore from Theorems B and D we get the following well-known result.

Corollary 1.15. *(See Gaschütz [13].) $\Delta^{\mathfrak{N}}(G)/\Phi(G) = Z(G/\Phi(G))$.*

Corollary 1.15 was a motivation for the following

Corollary 1.16. *For any solubly saturated* (*Shemetkov* [28]), *in particular, for any saturated* (*Selkin* [27]) *formation* $\mathfrak{F}$ *we have*

$$\Delta^{\mathfrak{F}}(G)/\Phi(G) = Z_{\mathfrak{F}}\big(G/\Phi(G)\big).$$

We use $Z^*(G)$ to denote the *quasicenter* of G, that is, the largest normal subgroup of G of the form $A \times A_1 \times \cdots \times A_t$, where $A \leqslant Z(G)$ and A_i is a normal simple non-abelian subgroup of G such that $G = A_i C_G(A_i)$ $(i = 1, \ldots, t)$.

Corollary 1.15 is also a motivation for the following result.

Corollary 1.17. $\Delta^{\mathfrak{N}^*}(G)/\Phi(G) = Z^*(G/\Phi(G))$.

Proof. Let $\Delta = \Delta^{\mathfrak{N}^*}(G)$. In view of Theorems B and D, $\Delta/\Phi(G) = Z_{\mathfrak{N}^*}(G/\Phi(G))$. Hence $\Delta/\Phi(G)$ is a normal quasinilpotent subgroup of $G/\Phi(G)$ by Theorem C(i). Since $\Phi(G/\Phi) = 1$, it follows (see the remark before Corollary 2.15) that $\Delta/\Phi(G) = (N_1/\Phi(G)) \times \cdots \times (N_t/\Phi(G))$, where $N_i/\Phi(G)$ is a minimal normal subgroup of $G/\Phi(G)$. Now, the corollary follows from the well-known fact (see Theorem 13.6 in [19, Chapter X]) that a chief factor H/K of G is $\mathfrak{N}^*$-central in G if and only if H/K is simple and $G/K = (H/K)(C_G(H/K))$. $\square$

From Theorems C(ii) and D we also get

Corollary 1.18. *(See Theorem 8.12 in [30] or Selkin [27].) Let $\mathfrak{F}$ be a normally hereditary saturated formation containing all nilpotent groups and E a normal subgroup of G. If $E/E \cap \Delta^{\mathfrak{F}}(G) \in \mathfrak{F}$, then $E \in \mathfrak{F}$.*

Finally, consider some applications of $\mathfrak{F}\Phi^*$-hypercentral subgroups in the theory of generalized S-quasinormal subgroups.

A subgroup H of G is said to be *subnormally embedded* in G if every Sylow subgroup of H is also a Sylow subgroup of some subnormal subgroup of G.

Recall that a subgroup A of a group G is said to be S-*quasinormal*, S-*permutable*, or $\pi(G)$-*permutable* in G (Kegel [20]) if $AP = PA$ for all Sylow subgroups P of G. Note that every S-quasinormal subgroup is subnormal (Kegel [20]). A subgroup H of a group G is said to be S-*quasinormally embedded* or S-*permutably embedded* in G (Ballester-Bolinches and Pedraza-Aguilera [4]) if each Sylow subgroup of H is also a Sylow subgroup of some S-quasinormal subgroup of G. Since every S-quasinormal subgroup is subnormal [20], every S-quasinormally embedded subgroup is subnormally embedded. On the other hand, from the properties of S-quasinormal subgroups (see Chapter 1 in [5]) it is easy to see that for any S-quasinormal p-group H of G and for any chief factor L/K of G the index $|G : N_G(K(H \cap L))|$ is a power of p. This circumstance is the main motivation for introducing the following generalizations of S-quasinormality.

Definition 1.19. Let H be a p-subgroup of G. We say that:

(i) *H satisfies Φ^*-property in G* if H is subnormally embedded in G and for any non-solubly-Frattini chief factor L/K of G, $|G : N_G(K(H \cap L))|$ is a power of p.
(ii) *H is $\mathfrak{F}_{\Phi^*}$-supplemented in G* if for some subgroups $T \leqslant G$ and $S \leqslant H$, where S satisfies Φ^*-property in G, we have that $HT = G$ and $H \cap T \subseteq SZ_{\mathfrak{F}}(T)$.

The following example shows that in general the set of all subgroups of G having Φ^*-property in G is wider than the set of all S-quasinormally embedded subgroups.

Example 1.20. Let V be a simple $\mathbb{F}_3 A_4$-module which is faithful for the alternating group A_4. Let $E = V \rtimes A_4$. Let $A = A_3(E)$ be the 3-Frattini module of E (see [10] or [7, p. 853]), and let D be a non-splitting extension of A by E. By Corollary 1 in [10], $V = Z = O_{3',3}(E) = C_E(A/Rad(A))$. Hence for some chief factor A/N of D we have $A/N \leqslant \Phi(D/N)$ and $D/C_D(A/N) \simeq A_4$. Thus $|A/N| > 3$. Let $G = (D/N) \times A_5$. Let L be a subgroup of order 3 of A/N such that L is contained in the center of a Sylow 3-subgroup G_3 of G, C a subgroup of order 5 in A_5 and $H = LC$. We shall show that L is not S-quasinormally embedded in G, L satisfies Φ^*-property in G and H is $\mathfrak{F}_{\Phi^*}$-supplemented G for any class $\mathfrak{F}$ of groups. Indeed, suppose that L is S-quasinormally embedded in G. Let W be an S-quasinormal subgroup of G such that L is a Sylow 3-subgroup of W. Then $L = (A/N) \cap W$ is S-quasinormal in G since the set of all S-quasinormal subgroups forms a sublattice of the lattice of all subgroups of G (Kegel [20]). Therefore $O^3(G) \leqslant N_G(L)$ by [5, Chapter 1, Lemma 1.2.16]. But then $G = G_3O^3(G) \leqslant N_G(L)$. Hence $A/N = L$, a contradiction. Therefore L is not S-quasinormally embedded in G. It is clear that L is subnormal in G. Now let H/K be any non-solubly-Frattini chief factor of G. Then H/K is non-Frattini, so there is a maximal subgroup M of G such that $K \leqslant M$ and $HM = G$. Suppose that $L \leqslant H$. Then $L \leqslant H \cap \Phi(G) \leqslant H \cap M = K$. Hence $L = L \cap H \leqslant K$, so L satisfies the Φ^*-property in G.

Finally, we show that H is $\mathfrak{F}_{\Phi^*}$-supplemented G for any class $\mathfrak{F}$ of groups. Let $T = (D/N)M$, where M is a complement of C in A_5. Then $G = HT$. Moreover, $H \cap (D/N)M = L(C \cap (D/N)M) = L$. Therefore H is $\mathfrak{F}_{\Phi^*}$-supplemented G.

Theorem E. *Let $\mathfrak{F}$ be a solubly saturated formation containing all supersoluble groups, and let $X \leqslant E$ be normal subgroups of G with $G/E \in \mathfrak{F}$. Suppose that every maximal subgroup of every non-cyclic Sylow subgroup of X is $\mathfrak{U}_{\Phi^*}$-supplemented in G. If X is either E or $F^*(E)$, then $G \in \mathfrak{F}$.*

The proof of Theorem E relies on Theorem A and the following two results, which generalize a large number of known results (see Section 5 in [34]) and, in particular, some recent results in [15, 18,22–25,38,39] (see Section 4).

Theorem 1.21. *Let $\mathfrak{F}$ be a solubly saturated formation containing all supersoluble groups, E a normal subgroup of G.*

(a) *If E is a p-subgroup of G and every maximal subgroup H of E is $\mathfrak{F}_{\Phi^*}$-supplemented in G, then E is $\mathfrak{F}\Phi^*$-hypercentral in G.*
(b) *If every maximal subgroup of every non-cyclic Sylow subgroup of E is $\mathfrak{U}_{\Phi^*}$-supplemented in G, then E is $\mathfrak{U}\Phi^*$-hypercentral in G.*

Theorem 1.22. *Let $\mathfrak{F}$ be the class of all p-nilpotent groups and E a normal subgroup of G such that p divides $|E|$ and $(p-1, |E|) = 1$. Let P be a Sylow p-subgroup of E. If every maximal subgroup of P is $\mathfrak{F}_{\Phi^*}$-supplemented in G, then E is p-nilpotent and $E/O_{p'}(E)$ is $\mathfrak{U}\Phi^*$-hypercentral in $G/O_{p'}(E)$.*

All unexplained notation and terminology are standard. The reader is referred to [3,7] or [14] if necessary.

2. Proofs of Theorems A, B, C and D

Lemma 2.1. *Let N and K be subgroups of G, where N is normal and K is subnormal in G.*

(a) *$\Phi^*(K) \leqslant \Phi^*(G)$.*
(b) *$\Phi^*(G)N/N \leqslant \Phi^*(G/N)$.*
(c) *If $N \leqslant \Phi^*(G)$, then $\Phi^*(G/N) = \Phi^*(G)/N$.*
(d) *If $G = G_1 \times \cdots \times G_t$, then $\Phi^*(G) = \Phi^*(G_1) \times \cdots \times \Phi^*(G_t)$.*
(e) *Let H/K be a solubly Frattini chief factor of G. If $HN \neq KN$, then HN/KN is a solubly Frattini chief factor of G/KN.*

Proof. (a) Let $K = K_t \trianglelefteq \cdots \trianglelefteq K_0 = G$. We prove the assertion by induction on t. Since $R(K_1)$ is characteristic in K_1, it is normal in G. Hence

$$\Phi^*(K_1) = \Phi\bigl(R(K_1)\bigr) \leqslant \Phi\bigl(R(G)\bigr) = \Phi^*(G)$$

by [7, Chapter A, Theorem 9.2(d)]. On the other hand, $\Phi^*(K) \leqslant \Phi^*(K_1)$ by induction, so $\Phi^*(K) \leqslant \Phi^*(G)$.

(b) Let $f : R(G)/R(G) \cap N \to R(G)N/N$ be the canonical isomorphism from $R(G)/R(G) \cap N$ onto $R(G)N/N$. Then $f(\Phi(R(G)/R(G) \cap N)) = \Phi(R(G)N/N)$ and

$$f\bigl(\Phi\bigl(R(G)\bigr)\bigl(R(G) \cap N\bigr)/\bigl(R(G) \cap N\bigr)\bigr) = \Phi\bigl(R(G)\bigr)N/N.$$

But by [7, Chapter A, Lemma 9.2(e)] we have

$$\Phi\bigl(R(G)\bigr)\bigl(R(G) \cap N\bigr)/\bigl(R(G) \cap N\bigr) \leqslant \Phi\bigl(R(G)/R(G) \cap N\bigr).$$

Then, since $R(G)N/N \leqslant R(G/N)$, we have

$$\Phi^*(G)N/N = \Phi\bigl(R(G)\bigr)N/N \leqslant \Phi\bigl(R(G)N/N\bigr) \leqslant \Phi\bigl(R(G/N)\bigr) = \Phi^*(G/N)$$

by (a).

(c) Since $N \leqslant \Phi^*(G) = \Phi(R(G))$, N is nilpotent, and so $R(G)/N = R(G/N)$. Hence

$$\Phi^*(G)/N = \Phi\bigl(R(G)\bigr)/N = \Phi\bigl(R(G/N)\bigr) = \Phi^*(G/N).$$

(d) This follows from Theorem 9.4 in [7, Chapter A] and the fact that $R(G) = R(G_1) \times \cdots \times R(G_t)$.

(e) By (b),

$$(KN/K)(H/K)/(KN/K) = (NH/K)/(KN/K) \leqslant \Phi^*\bigl((G/K)/(KN/K)\bigr).$$

Hence from the G-isomorphism $(G/K)/(KN/K) \simeq G/KN$ we get that $NH/KN \leqslant \Phi^*(G/KN)$. Finally, in view of the G-isomorphism $HN/KN \simeq H/K(H \cap N)$, HN/KN is a chief factor of G.

The lemma is proved. □

The following lemma is obvious.

Lemma 2.2. *Let H/K and E/T be chief factors of a group G. If H/K and E/T are G-isomorphic, then $(H/K) \rtimes (G/C_G(H/K)) \simeq (E/T) \rtimes (G/C_G(E/T))$.*

The following lemma shows that $Z_{\mathfrak{F}\Phi^*}(G)$ is the largest normal $\mathfrak{F}\Phi^*$-hypercentral subgroup of G.

Lemma 2.3. *Let $\mathfrak{F}$ be a formation, A and B normal subgroups of G.*

(a) *If A is $\mathfrak{F}\Phi^*$-hypercentral in G, then AB/B is $\mathfrak{F}\Phi^*$-hypercentral in G/B.*
(b) *A and B are $\mathfrak{F}\Phi^*$-hypercentral in G, then AB is $\mathfrak{F}\Phi^*$-hypercentral in G.*
(c) *$Z_{\mathfrak{F}\Phi^*}(G)$ is $\mathfrak{F}\Phi^*$-hypercentral in G.*
(d) *$Z_{\mathfrak{F}\Phi^*}(G)A/A \leqslant Z_{\mathfrak{F}\Phi^*}(G/A)$.*
(e) *If A is $\mathfrak{F}\Phi^*$-hypercentral in G, then*

$$Z_{\mathfrak{F}\Phi^*}(G)/A = Z_{\mathfrak{F}\Phi^*}(G/A).$$

(f) *$Z_{\mathfrak{F}\Phi^*}(G/Z_{\mathfrak{F}\Phi^*}(G)) = 1$.*

Proof. (a) Let $1 = A_0 < A_1 < \cdots < A_t = A$ $(*)$ be a chief series of G below A such that every non-solubly-Frattini factor A_i/A_{i-1} of Series $(*)$ is $\mathfrak{F}$-central in G. Let H/K be a chief factor of G such that for some i we have $K = BA_{i-1}$ and $H = BA_i$. Then H/K is G-isomorphic to $A_i/A_{i-1}(A_i \cap B) = A_i/A_{i-1}$. Suppose that H/K is not solubly Frattini. Then, by Lemma 2.1(e), A_i/A_{i-1} is also not solubly Frattini. Hence A_i/A_{i-1} is $\mathfrak{F}$-central in G, so H/K $\mathfrak{F}$-central in G by Lemma 2.2. Therefore every non-solubly-Frattini factor of the series $1 \leqslant BA_0/B \leqslant BA_1/B \leqslant \cdots \leqslant BA_t/B = BA/B$ is $\mathfrak{F}$-central in G/B.

(b) This follows from (a).

(c) This follows from (b).

(d) This follows from (a) and (b).

(e) By (d), $Z_{\mathfrak{F}\Phi^*}(G)/A \leqslant Z_{\mathfrak{F}\Phi^*}(G/A)$. On the other hand, for any $\mathfrak{F}\Phi^*$-hypercentral subgroup V/A of G/A the subgroup V is $\mathfrak{F}\Phi^*$-hypercentral in G since A is $\mathfrak{F}\Phi^*$-hypercentral in G. Hence $Z_{\mathfrak{F}\Phi^*}(G/A) \leqslant Z_{\mathfrak{F}\Phi^*}(G)/A$, so we have (e).

(f) This follows from (c) and (e).

The lemma is proved. □

Lemma 2.4. *Let N_1 and N_2 be distinct abelian minimal normal subgroups of G. Then there exists a bijection*

$$f : \{N_1, N_1N_2/N_1\} \to \{N_2, N_2N_1/N_2\}$$

such that corresponding chief factors are G-isomorphic and solubly Frattini chief factors correspond to one another.

Proof. Let $N = N_1 \times N_2$. Arguing as in the proof of Lemma 9.12 in [7, Chapter A], we need only consider the case where $N_1 \cap \Phi^*(G) = N_2 \cap \Phi^*(G) = 1$ and (say) $N/N_2 \leqslant \Phi(R(G/N_2))$. Since N_2 is abelian, $N_2 \leqslant R(G)$ and so $\Phi(R(G/N_2)) = \Phi(R(G)/N_2)$. Hence, by Lemma 9.11 in [7, Chapter A], $N \leqslant \Phi(R(G))N_2$, which implies that $N = N_2(N \cap \Phi(R(G))) = N_2(N \cap \Phi^*(G))$. Let $N_3 = N \cap \Phi^*(G)$. Then in view of the G-isomorphisms $N/N_1 = N_3N_1/N_1 \simeq N_3$ and $N/N_2 = N_3N_2/N_2 \simeq N_3$ we deduce from Lemma 2.1(b) that the map h with $h(N_1) = N_2$ and $h(N/N_1) = N/N_2$ is desired. □

Lemma 2.5. *Let H be a soluble normal subgroups of G. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be chief series of G below H. Then there exists a one-to-one correspondence between the chief factors of $\mathcal{H}_1$ and those of $\mathcal{H}_2$ such that corresponding factors are G-isomorphic and the non-solubly-Frattini chief factors of $\mathcal{H}_1$ correspond to the non-solubly-Frattini chief factors of $\mathcal{H}_2$.*

Proof. The assertion may be proved on the basis of Lemma 2.4 similarly to Theorem 9.13 in [7, A]. □

A group is called semisimple provided it is either identity or the direct product of some simple non-abelian groups.

Lemma 2.6. *Let N be a non-abelian minimal normal subgroup of G and $C = C_G(N)$. Let E be a normal quasinilpotent subgroup of G. If $N \leqslant E$, then $E = N \times (C \cap E)$.*

Proof. Let $F = F(E)$. Since F is characteristic in E, it is normal in G. Hence NF/F is G-isomorphic to N, and so $C = C_G(NF/F)$. On the other hand, by [7, Chapter A, Theorem 10.6(b)], $F \leqslant C$, which implies that $C_{G/F}(NF/F) = C/F$. By [19, Chapter X, Theorem 13.6], E/F is semisimple. Hence

$$E/F = \big(C_{G/F}(NF/F) \cap (E/F)\big)(NF/F) = \big((C/F) \cap (E/F)\big)(NF/F) = (C \cap E)N/F.$$

This implies that $E = N \times (C \cap E)$. □

For any formation function

$$f:\mathbb{P}\to\{\text{group formations}\},$$

the symbol $LF(f)$ denotes the collection of all groups G such that either $G=1$ or $G\neq 1$ and $G/C_G(H/K)\in f(p)$ for every chief factor H/K of G and every $p\in\pi(H/K)$.

In the following lemma, the symbol $\mathfrak{G}_pF(p)$ denotes that set of all groups A such that $A^{F(p)}$ is a p-group.

Lemma 2.7. *(See Proposition 3.8 in [7, Chapter IV] or Corollary 3.1.17 in [14].) For any non-empty saturated formation $\mathfrak{F}$, there is a unique formation function F such that $\mathfrak{F}=LF(F)$ and $F(p)=\mathfrak{G}_pF(p)\subseteq\mathfrak{F}$ for all primes p.*

The formation function F in Lemma 2.7 is called the canonical local satellite of $\mathfrak{F}$.

For any function f of the form

$$f:\mathbb{P}\cup\{0\}\to\{\text{group formations}\},\tag{$*$}$$

we put, following [36],

$$CF(f)=\bigl\{G\text{ is a group}\bigm| G/C_G(H/K)\in f(0)\text{ for each non-abelian chief factor }H/K\text{ of }G$$
$$\text{and }G/C_G(H/K)\in f(p)\text{ for any abelian }p\text{-chief factor }H/K\text{ of }G\bigr\}.$$

In the paper [36] the following useful facts are proved.

Lemma 2.8.

(1) *For any function f of the form $(*)$, the class $CF(f)$ is a solubly saturated formation.*
(2) *For any non-empty solubly saturated formation $\mathfrak{F}$, there is a unique function F of the form $(*)$ such that $\mathfrak{F}=CF(F)$, $F(p)=\mathfrak{G}_pF(p)\subseteq\mathfrak{F}$ for all primes p, and $F(0)=\mathfrak{F}$.*
(3) *If $\mathfrak{F}=LF(H)$ is a saturated formation, where H is the canonical local satellite of $\mathfrak{F}$, then $\mathfrak{F}=CF(F)$, where $F(p)=H(p)$ for all primes p.*

The function F in Lemma 2.8 is called the canonical composition satellite of $\mathfrak{F}$.

Lemma 2.9. *(See [3, Chapter 2, Proposition 3.1.40].) If $\mathfrak{F}$ is a normally hereditary (solubly) saturated formation and F is the canonical local (the canonical composition, respectively) satellite of $\mathfrak{F}$, then $F(p)$ is normally hereditary for all primes p.*

Lemma 2.10. *Suppose that $\mathfrak{F}$ is a solubly saturated formation containing all nilpotent groups, $N\leqslant E$ and $E/N\in\mathfrak{F}$. If either $N\leqslant\Phi^*(G)$ [31] or $\mathfrak{F}$ is saturated and $N\leqslant\Phi(G)$ [28], then $E\in\mathfrak{F}$.*

Lemma 2.11. *(See Lemma 2.11 in [35].) $F^*(G)\leqslant C_G(H/K)$ for any abelian chief factor H/K of G.*

Lemma 2.12. *(See Lemma 12.8 in [32].) If H/K is a chief factor of G and M is a maximal subgroup of G such that $K\leqslant M$ and $MH=G$, then*

$$G/M_G\simeq(HM_G/M_G)\rtimes\bigl(G/C_G(HM_G/M_G)\bigr)\simeq(H/K)\rtimes\bigl(G/C_G(H/K)\bigr).$$

Lemma 2.13. *Let $\mathfrak{F} = CF(F)$ be a (solubly) saturated formation, where F is the canonical local (the canonical composition, respectively) satellite of $\mathfrak{F}$. Let H/K be a chief factor of G. Then H/K is $\mathfrak{F}$-central in G if and only if $G/C_G(H/K) \in F(p)$ in the case where H/K is a p-group, and $G/C_G(H/K) \in \mathfrak{F}$ in the case where H/K is non-abelian.*

Proof. Without loss we can assume that $K = 1$. If H is abelian, then $C_{H \rtimes (G/C_G(H))}(H) = H$, so the assertion follows from Lemma 2.8(2), (3). Finally, assume that H is non-abelian. Then $C_G(H) \cap H = 1$, so in view of Lemma 2.2 and the G-isomorphism $C_G(H)H/C_G(H) \simeq H$ we may assume that $C_G(H) = 1$. Therefore we have only to show that if $G \simeq G/C_G(H) \in \mathfrak{F}$, then $H \rtimes G \in \mathfrak{F}$. Let M be a maximal subgroup of G such that $HM = G$. Then $M_G \leqslant C_G(H) = 1$, so $H \rtimes G \simeq G$ by Lemma 2.12.

The following lemma is a corollary of general results on f-hypercentral action (see [7, Chapter IV, Section 6]). For reader's convenience, we give a direct proof. $\square$

Lemma 2.14. *Let $\mathfrak{F}$ be a saturated (solubly saturated) formation and F the canonical local (the canonical composition, respectively) satellite of $\mathfrak{F}$. Let E be a normal p-subgroup of G. Then $E \leqslant Z_{\mathfrak{F}}(G)$ if and only if $G/C_G(E) \in F(p)$.*

Proof. First suppose that $\mathfrak{F} = CF(F)$ is solubly saturated. Let $1 = E_0 < E_1 < \cdots < E_t = E$ be a chief series of G below E. Let $C_i = C_G(E_i/E_{i-1})$ and $C = C_1 \cap \cdots \cap C_t$. Then $C_G(E) \leqslant C$ and so $C/C_G(E)$ is a p-group by Corollary 3.3 in [9, Chapter 5]. If $E \leqslant Z_{\mathfrak{F}}(G)$, then by Lemma 2.13, $G/C_i \in F(p)$ for all $i = 1, \dots, t$, so $G/C \in F(p)$. This induces that $G/C_G(E) \in F(p) = \mathfrak{G}_p F(p)$. On the other hand, if $G/C_G(E) \in F(p)$, then $G/C_i \in F(p)$ for all $i = 1, \dots, t$, so $E \leqslant Z_{\mathfrak{F}}(G)$ by Lemma 2.13.

Now let $\mathfrak{F} = LF(F)$ be a saturated formation. Then $\mathfrak{F}$ is solubly saturated, and if H is the canonical composition satellite of $\mathfrak{F}$, then $H(p) = F(p)$ for all primes p by Lemma 2.8(3). Hence the second assertion of the lemma is a corollary of the first one. The lemma is proved. $\square$

Lemma 2.15. *Let E be a normal non-identity quasinilpotent subgroup of G. If $\Phi(G) \cap E = 1$, then E is the direct product of some minimal normal subgroups of G.*

Proof. Let N be a minimal normal subgroup of G contained in E and $C = C_G(N)$. First we show that for some normal subgroup D of G we have $E = N \times D$. If N is non-abelian, then $E = N \times (E \cap C)$ by Lemma 2.6. So in this case we may take $D = E \cap C$. Now suppose that N is an abelian group. Since $\Phi(G) \cap E = 1$, for some maximal subgroup M of G we have $G = N \rtimes M$ and so $E = N \rtimes (E \cap M)$. Let $D = E \cap M$. Note that $N \leqslant F(E) \leqslant Z_\infty(E)$ by [19, Chapter X, Theorem 13.6]. On the other hand, N is the direct product of some minimal normal subgroup of E by [7, Chapter A, Proposition 4.13(c)], so $N \leqslant Z(E)$. Therefore $G = NM \leqslant N_G(E \cap M)$. Thus $E = N \times D$, where D is normal in G. If $D \neq 1$, then, by induction, D is the product of some minimal normal subgroups of G. Hence E is also the product of some minimal normal subgroups of G. $\square$

Proof of Theorem A. (i) Suppose that this assertion is false and consider a counterexample (G, E) for which $|G| + |E|$ is minimal. Put $F^* = F^*(E)$ and $F = F(E)$.

Let $1 = N_0 < N_1 < \cdots < N_t = F^*$ be a chief series of G below F^* such that every non-solubly-Frattini factor N_i/N_{i-1} of this series is $\mathfrak{F}$-central in G. Let $N = N_1$ and $C = C_E(N)$. Note that in the case where N is $\mathfrak{F}$-central in G we have $G/C_G(N) \in \mathfrak{F}$. Hence every chief factor of G between $C_G(N)$ and G is $\mathfrak{F}$-central in G. Hence from the G-isomorphism $E/C \simeq C_G(N)E/C_G(N)$ we deduce that every chief factor of G between C and E is $\mathfrak{F}$-central in G.

Suppose N is non-abelian. Let $D = N \times C$. Since $C = C_G(N) \cap E$, D is normal in G. Moreover, $F^* = F^*(D)$. Indeed, in view of Lemma 2.6, $F^* \leqslant F^*(D)$. On the other hand, $F^*(D) \leqslant F^*$ by Theorem 13.10 in [19, Chapter X]. Hence $F^* = F^*(D)$ and so the hypothesis holds for D. Suppose that $D \neq E$. Then D is $\mathfrak{F}\Phi^*$-hypercentral in G by the choice of $|G| + |E|$. On the other hand, from the above we know that E/D is $\mathfrak{F}$-hypercentral in G/D. Hence E is $\mathfrak{F}\Phi^*$-hypercentral in G, a contradiction. Therefore $D = E$. We now show that in this case the hypothesis holds for $(G/N, E/N)$. Let $W/N = F^*(E/N)$. In view of Lemma 13.3(a) in [19, Chapter X], $F^*/N \leqslant W/N$. On the other hand,

$W = N \times (W \cap C)$, where $W \cap C \simeq W/N$ is quasinilpotent, so W is quasinilpotent by Lemma 13.3(d) in [19, Chapter X]. Hence $F^*/N = W/N$. This implies that $F^*(E/N)$ is $\mathfrak{F}\Phi^*$-hypercentral in G/N by Lemma 2.3(a). The choice of $|G| + |E|$ implies that E/N is $\mathfrak{F}\Phi^*$-hypercentral in G/N and so E is $\mathfrak{F}\Phi^*$-hypercentral in G since N is $\mathfrak{F}$-central in G. This contradiction shows that N is abelian.

Assume that $N \leqslant \Phi^*(G)$. Then in view of Lemma 2.10, $F^*(E/N) = F^*(E)/N$ and so E/N is $\mathfrak{F}\Phi^*$-hypercentral in G/N by the choice of (G, E), which implies that E is $\mathfrak{F}\Phi^*$-hypercentral in G, a contradiction. Hence $N \nleqslant \Phi^*(G)$, and consequently N is $\mathfrak{F}$-central in G. Now we show that the hypothesis is still true for $(G/N, C/N)$. Indeed, clearly $N \leqslant Z(C)$. Moreover, $F^* \leqslant C$ by Lemma 2.11. Therefore $F^*(C/N) = F^*/N$ by [19, X, Theorem 13.6]. This shows that the hypothesis is still true for $(G/N, C/N)$. The choice of (G, E) implies that C/N is $\mathfrak{F}\Phi^*$-hypercentral in G/N. Then since N is $\mathfrak{F}$-central in G, C is $\mathfrak{F}\Phi^*$-hypercentral in G. It follows that E is $\mathfrak{F}\Phi^*$-hypercentral in G. This contradiction completes the proof of (i).

(ii) See the proof of (i).

(iii) See the proof of (i) and use Lemma 2.3 in [33], which is an analog of Lemma 2.3. □

Proof of Theorem B. It is clear that $\Phi(G) \leqslant Z_{\mathfrak{F}\Phi}(G)$. By Lemma 2.3 in [33] every non-Frattini chief factor of G below $Z_{\mathfrak{F}\Phi}(G)$ is $\mathfrak{F}$-central in G. Hence we have $Z_{\mathfrak{F}}(G/\Phi(G)) \leqslant Z_{\mathfrak{F}\Phi}(G/\Phi(G)) = Z_{\mathfrak{F}\Phi}(G)/\Phi(G)$.

Now let $L/\Phi(G)$ be a minimal normal subgroup of $G/\Phi(G)$ contained in $Z_{\mathfrak{F}\Phi}(G/\Phi(G))$. Then $L/\Phi(G)$ is non-Frattini. Hence this chief factor of G is $\mathfrak{F}$-central in G. Thus $\mathrm{Soc}(Z_{\mathfrak{F}\Phi}(G/\Phi(G))) \leqslant Z_{\mathfrak{F}}(G/\Phi(G))$. On the other hand, since $F^*(Z_{\mathfrak{F}\Phi}(G/\Phi(G)))$ is characteristic in $Z_{\mathfrak{F}\Phi}(G/\Phi(G))$, it is normal in $G/\Phi(G)$. Hence in view of Lemma 2.15, $F^*(Z_{\mathfrak{F}\Phi}(G/\Phi(G)))$ is the product of some minimal normal subgroups of $G/\Phi(G)$. But then $F^*(Z_{\mathfrak{F}\Phi}(G/\Phi(G))) \leqslant Z_{\mathfrak{F}}(G/\Phi(G))$. Hence $Z_{\mathfrak{F}\Phi}(G/\Phi(G)) \leqslant Z_{\mathfrak{F}}(G/\Phi(G))$ by Theorem A(ii), which implies that

$$Z_{\mathfrak{F}\Phi}(G)/\Phi(G) = Z_{\mathfrak{F}}\big(G/\Phi(G)\big).$$

It follows that $Z_{\mathfrak{F}\Phi}(G)/\Phi(G)$ is a π-group. Hence $Z_{\mathfrak{F}\Phi}(G)/O_{\pi'}(\Phi(G))$ is a π-group. By the Schur–Zassenhaus theorem the subgroup $O_{\pi'}(\Phi(G))$ has a complement A in $Z_{\mathfrak{F}\Phi}(G)$ and any two complements of $O_{\pi'}(\Phi(G))$ in $Z_{\mathfrak{F}\Phi}(G)$ are conjugate in $Z_{\mathfrak{F}\Phi}(G)$. Then by the Frattini Argument, $G = Z_{\mathfrak{F}\Phi}(G)N_G(A) = O_{\pi'}(\Phi(G))N_G(A) = N_G(A)$. Therefore $Z_{\mathfrak{F}\Phi}(G) = A \times O_{\pi'}(\Phi(G))$ and $A = O_{\pi}(Z_{\mathfrak{F}\Phi}(G))$. Moreover, in view of the G-isomorphism $A\Phi(G)/\Phi(G) \simeq A/A \cap \Phi(G)$ from the above we deduce that $A/A \cap \Phi(G) \leqslant Z_{\mathfrak{F}}(G/A \cap \Phi(G))$. The theorem is proved. □

Proof of Theorem C. (i) Suppose that this assertion is false and consider a counterexample (G, E) for which $|G| + |E|$ is minimal. Let $K = E \cap Z_{\mathfrak{F}\Phi^*}(G)$. Then $K \neq 1$, in particular $Z_{\mathfrak{F}\Phi^*}(G) \neq 1$. By Lemma 2.3(c), $Z_{\mathfrak{F}\Phi^*}(G)$ is $\mathfrak{F}\Phi^*$-hypercentral in G. Hence there is a minimal normal subgroup L of G such that either $L \leqslant \Phi^*(G)$ or L is $\mathfrak{F}$-central in G.

We shall show that for any minimal normal subgroup N of G we have $N \leqslant E$ and $E/N \in \mathfrak{F}$. Indeed, first note that $KN/N \leqslant EN/N$ and

$$(EN/N)/(KN/N) \simeq EN/KN \simeq E/K(E \cap N) \simeq (E/K)/\big(K(E \cap N)/K\big) \in \mathfrak{F}.$$

On the other hand, by Lemma 2.3(d), $KN/N \leqslant Z_{\mathfrak{F}\Phi^*}(G/N)$. Therefore

$$(EN/N)/\big((EN/N) \cap Z_{\mathfrak{F}\Phi^*}(G/N)\big) \in \mathfrak{F}.$$

The choice of (G, E) implies that $EN/N \in \mathfrak{F}$. If $N \nleqslant E$, then from the isomorphism $E \simeq EN/N$ we deduce $E \in \mathfrak{F}$, which contradicts the choice of (G, E). Hence $N \leqslant E$. It follows that $N = L$ is the unique minimal normal subgroup of G and $L \nleqslant \Phi^*(G)$, otherwise $E \in \mathfrak{F}$ by Lemma 2.10. Thus L is $\mathfrak{F}$-central in G.

Let $C = C_G(L)$. Then $L \rtimes (G/C) \in \mathfrak{F}$. Suppose that L is non-abelian. Then $C = 1$. Since $\mathfrak{F}$ is normally hereditary and $G \simeq G/C = G/1 \in \mathfrak{F}$, $E \in \mathfrak{F}$. This contradiction shows that L is a p-group for some prime p. In view of Lemma 2.8(2) and Lemma 2.9, $\mathfrak{F} = CF(F)$, where F is the canonical composition satellite of $\mathfrak{F}$ and $F(p)$ is a normally hereditary formation. In view of Lemma 2.13, $G/C \in F(p)$, hence $E/C \cap E \simeq CE/C \in F(p)$. Thus $L \leqslant Z_{\mathfrak{F}}(E)$ by Lemma 2.14. Since $E/L \in \mathfrak{F}$, it follows that $E \in \mathfrak{F}$. This contradiction completes the proof of (i).

(ii) See the proof of (i) and use Lemma 2.3 in [33].

(iii) Let L/D be a minimal normal subgroup of E/D. If $L/D \leqslant \Phi(E/D)$, then $L/D \leqslant \Phi(G/D)$ and so $L \leqslant E \cap Z_{\mathfrak{F}\Phi}(G) = D$, a contradiction. Hence $L/D \nleqslant \Phi(E/D)$, which implies that $\Phi(E/D) = 1$. But $\mathrm{Soc}(E/D) \leqslant Z_{\mathfrak{F}\Phi}(E/D)$, so $\mathrm{Soc}(E/D) \leqslant Z_{\mathfrak{F}}(E/D)$. Since $\Phi(E/D) = 1$, we also see by Lemma 2.15 that $F^*(E/D)$ is the direct product of some minimal normal subgroups of E/D, so $F^*(E/D) \leqslant Z_{\mathfrak{F}}(E/D)$. Therefore $E/D = Z_{\mathfrak{F}}(E/D)$ by Theorem A(ii) and hence $E/D \in \mathfrak{F}$. Now, arguing as in the proof of (i), one can show that $E \in \mathfrak{F}$. □

Proof of Theorem D. First note that if M is a maximal subgroup of G and H/K is a chief factor H/K of G such that $K \leqslant M$ and $H \nleqslant M$, then by Lemma 2.12, M is $\mathfrak{F}$-abnormal in G if and only if H/K is $\mathfrak{F}$-eccentric in G.

Assume that for some maximal $\mathfrak{F}$-abnormal subgroup M of G we have $Z_{\mathfrak{F}\Phi}(G) \nleqslant M$. Since by Theorem B,

$$Z_{\mathfrak{F}\Phi}(G)/\Phi(G) = Z_{\mathfrak{F}}\big(G/\Phi(G)\big),$$

there is a chief factor H/K of G such that $\Phi(G) \leqslant K \leqslant M$, $H \nleqslant M$ and $H/\Phi(G) \leqslant Z_{\mathfrak{F}}(G/\Phi(G))$. But then H/K is $\mathfrak{F}$-central in G, which contradicts the $\mathfrak{F}$-abnormality of M. This contradiction shows that $Z_{\mathfrak{F}\Phi}(G) \leqslant \Delta^{\mathfrak{F}}(G)$. If the inverse inclusion is not true, then there is a non-Frattini chief factor H/K of G such that $H \leqslant \Delta^{\mathfrak{F}}(G)$ and H/K is not $\mathfrak{F}$-central. Let M be a maximal subgroup of G such that $K \leqslant M$ and $H \nleqslant M$. Then M is $\mathfrak{F}$-abnormal in G, so $H \leqslant \Delta^{\mathfrak{F}}(G) \leqslant M$, a contradiction. Thus $\Delta^{\mathfrak{F}}(G) \leqslant Z_{\mathfrak{F}\Phi}(G)$. The theorem is proved. □

3. The proof of Theorems E, 1.21 and 1.22

Lemma 3.1. *Let $L \leqslant G$ and p divides the order of L. Suppose that L is p-closed and $|G : N_G(L)|$ is a power of p. If for a Sylow p-subgroup L_p of L we have $L_p \leqslant V \lhd P$, where P is a Sylow p-subgroup of G, then $(L_p)^G \leqslant V$.*

Proof. Since L_p is characteristic in L, $N_G(L) \leqslant N_G(L_p)$. Hence $(L_p)^G = (L_p)^{N_G(L_p)P} = (L_p)^P \leqslant V$. □

Lemma 3.2. *Let $\mathfrak{F}$ be any non-empty class of groups, N a normal subgroup of G and $H \leqslant G$.*

(1) *If H satisfies Φ^*-property in G, then HN/N satisfies Φ^*-property in G/N.*
(2) *If H is $\mathfrak{F}_{\Phi^*}$-supplemented in G and either $N \leqslant H$ or $(|H|, |N|) = 1$, then HN/N is $\mathfrak{F}_{\Phi^*}$-supplemented in G/N.*

Proof. (1) It is clear.

(2) Suppose that H is $\mathfrak{F}_{\Phi^*}$-supplemented in G and let T be a subgroup of G such that $HT = G$ and $H \cap T \subseteq SZ_{\mathfrak{F}}(T)$, where $S \leqslant H$ and S satisfies Φ^*-property in G. Then $(NT/N) \cap (HN/N) = N(T \cap H)/N \subseteq (SN/N)(Z_{\mathfrak{F}}(T)N/N)$, where SN/N satisfies Φ^*-property in G/N by (1) and $Z_{\mathfrak{F}}(T)N/N \leqslant Z_{\mathfrak{F}}(T/N)$ by Lemma 2.2 in [17]. Hence HN/N is $\mathfrak{F}_{\Phi^*}$-supplemented in G/N. □

Note that if H/K is a cyclic chief factor of G, then $G/C_G(H/K)$ is cyclic. Hence $(H/K) \rtimes (G/C_G(H/K))$ is supersoluble. Therefore from Lemmas 2.8 and 2.13 we get

Lemma 3.3. *Let $\mathfrak{F}$ be a solubly saturated formation containing all supersoluble groups and E a normal subgroups of G with $G/E \in \mathfrak{F}$. If every chief factor of G below E is cyclic, then $G \in \mathfrak{F}$.*

In our proofs we shall need the following properties of subnormal subgroups.

Lemma 3.4. *Let G be a group, $A \leqslant K \leqslant G$ and $B \leqslant G$.*

(1) *Suppose that A is normal in G. Then K/A is subnormal in G/A if and only if K is subnormal in G* [7, A, 14.1].
(2) *If A is subnormal in G, then $A \cap B$ is subnormal in B* [7, A, 14.1].
(3) *If A is subnormal in G and A is a π-subgroup of G, then $A \leqslant O_\pi(G)$* [40].
(4) *If A is subnormal in G and B is a minimal normal subgroup of G, then $B \leqslant N_G(A)$* [7, A, 14.3].
(5) *If A and B are subnormal in G and $A = A'$ and $B = B'$, then $AB = BA$* [41].

The following lemma is evident.

Lemma 3.5. *Let A and B be subgroups of G such that $G = AB$. Then $A^x B = G$ for all $x \in G$.*

Recall that a group G is said to be π-closed if G has a normal Hall π-subgroup. A group G is said to be a C_π-group if G has a Hall π-subgroup and any two Hall π-subgroups of G are conjugate in G.

Lemma 3.6. *Let P be a p-subgroup of G. Suppose that G is a C_π-group for some set of primes π with $p \notin \pi$. Let V_1 and V_2 be maximal subgroups of P and T_i be a supplement of V_i in G. Suppose that $T_1 = N_G(H_1)$, where H_1 is a Hall π-subgroup of T_1, and that $T_1 \cap P \leqslant V_2$. If G is not π-closed, then T_2 is also not π-closed.*

Proof. Suppose that T_2 is π-closed. Without loss of generality we may assume that $T_2 = N_G(H_2)$, where H_2 is a Hall π-subgroup of T_2. Since $G = V_1T_1 = V_2T_2$ and $p \notin \pi$, H_1 and H_2 are Hall π-subgroups of G. By hypothesis, $(H_2)^x = H_1$ for some $x \in G$. Therefore, by Lemma 3.5, $G = V_2T_2 = V_2(T_2)^x = V_2T_1$ and so $P = P \cap V_2T_1 = V_2(P \cap T_1) = V_2$, a contradiction. The lemma is proved. □

Corollary 3.7. *Let P be a p-subgroup of G. Suppose that G is a C_π-group for some set of primes π with $p \notin \pi$. If every maximal subgroup of P has a π-closed supplement in G, then G is π-closed.*

Proof. Suppose that this assertion is false. Let V be a maximal subgroup of P and T a π-closed supplement of V in G. Without loss of generality, we may assume that $T = N_G(H)$, where H is a Hall π-subgroup of T. It is clear that $T \cap P < P$. Let W be a maximal subgroup of P such that $T \cap P \leqslant W$. Then by hypothesis, W has a π-closed supplement in G, which is impossible by Lemma 3.6. □

Lemma 3.8. *(See [11].) Suppose G has a Hall $2'$-subgroup. Then G is a $C_{2'}$-group.*

In the following, for an $\mathfrak{F}_{\Phi^*}$-supplemented subgroup H of G, we write Γ_H to denote the set of all triples (H, S, T), where $S \leqslant H$ and S satisfies Φ^*-property in G and T is a subgroup of G with $HT = G$ and $H \cap T \subseteq SZ_{\mathfrak{F}}(T)$.

Proof of Theorem 1.21(a). Suppose that this assertion is false and consider a counterexample (G, E) for which $|G| + |E|$ is minimal. Let $Z = Z_{\mathfrak{F}}(G)$. We proceed the proof via the following steps.

(1) *E is not $\mathfrak{U}\Phi^*$-hypercentral in G. In particular, $|E| > p$.* (This follows from the hypothesis that $\mathfrak{F}$ contains all supersoluble groups and the choice of (G, E).)
(2) *If N is a minimal normal subgroup of G contained in E, then E/N is $\mathfrak{F}\Phi^*$-hypercentral in G/N, N is the only minimal normal subgroup of G contained in E and $N/1$ is $\mathfrak{F}$-eccentric in G. In particular, $|N| > p$ and $E \cap Z = 1$.*

Indeed, by Lemma 3.2(2) the hypothesis holds for $(G/N, E/N)$ for any minimal normal subgroup N of G contained in E. Hence E/N is $\mathfrak{F}\Phi^*$-hypercentral in G/N by the choice of (G, E). This implies that N is both $\mathfrak{F}$-eccentric in G and not contained in $\Phi^*(G)$ by the choice of (G, E). In particular, $|N| > p$ since $\mathfrak{F}$ contains all supersoluble groups. Consequently, $N \nleqslant Z$. Suppose that G has two different minimal normal subgroups L and N contained in E. Then from the G-isomorphism $LN/N \simeq L$ it follows that $LN/N \leqslant \Phi^*(G/N)$. Let $D = LN$. Since N is abelian, $N \leqslant R(G)$ and so $\Phi(R(G/N)) = \Phi(R(G)/N)$. Hence, by Lemma 9.11 in [7, Chapter A], $D \leqslant \Phi(R(G))N$, which implies that $D = N(D \cap \Phi(R(G))) = N(D \cap \Phi^*(G))$. It follows that for some minimal normal subgroup R of G contained in E we have $R \leqslant \Phi^*(G)$, a contradiction. Hence N is the only minimal normal subgroup of G contained in E. Finally, since $E \cap Z$ is normal in G and $N \nleqslant Z$, we have $E \cap Z = 1$. Thus (2) holds.

(3) $\Phi(E) \neq 1$.

Suppose that $\Phi(E) = 1$. Then E is an elementary abelian p-group. Let W be a maximal subgroup of N such that W is normal in a Sylow p-subgroup of G. Then in view of (2), $W \neq 1$. Let B be a complement of N in E and $H = WB$. Then H is a maximal subgroup of E. Hence H is $\mathfrak{F}_{\Phi^*}$-supplemented in G. Let $(H, S, T) \in \Gamma_H$. Suppose that $T = G$. Then $H \leqslant SZ$. Hence $H = S(H \cap Z)$. But in view of (2), $H \cap Z \leqslant E \cap Z = 1$. Thus $H = S$ and thereby $|G : N_G(W)| = |G : N_G(S \cap N)|$ is a power of p. It follows that W is normal in G and consequently $W = 1$, a contradiction. Therefore $T \neq G$ and so $T \cap E$ is a non-identity normal subgroup of G. Hence $N \leqslant T$, which implies that $W \leqslant T \cap H \subseteq SZ_{\mathfrak{F}}(T) \cap E = S(Z_{\mathfrak{F}}(T) \cap E)$. We now show that $D = Z_{\mathfrak{F}}(T) \cap E = 1$. Indeed, suppose that $D \neq 1$ and let L be a minimal normal subgroup of G contained in D. By Lemma 2.8(2), $\mathfrak{F} = CF(F)$, where F is the canonical composition satellite of $\mathfrak{F}$. Then $T/C_T(L) \in F(p)$ by Lemma 2.14. Hence

$$\begin{aligned} G/C_G(L) &= ET/C_G(L) = ET/E\bigl(C_G(L) \cap T\bigr) \simeq T/E\bigl(C_G(L) \cap T\bigr) \cap T \\ &= T/\bigl(C_G(L) \cap T\bigr) = T/C_T(L) \in F(p). \end{aligned}$$

It follows that $L \leqslant Z$, which contradicts (2). Hence $D = 1$. This means that $W = WB \cap N = H \cap N = S \cap N$. Since S satisfies Φ^*-property, $|G : N_G(W)| = |G : N_G(S \cap N)|$ is a power of p. This shows that W is normal in G and consequently $|N| = p$, a contradiction. Hence (3) holds.

The final contradiction.

By (3), $\Phi(E) \neq 1$. Let N be a minimal normal subgroup of G contained in $\Phi(E)$. Then the hypothesis is still holds for $(G/N, E/N)$ by Lemma 3.2. Hence E/N is $\mathfrak{F}\Phi^*$-hypercentral in G/N by (2). But since $N \leqslant \Phi(E) \leqslant \Phi^*(G)$, E is $\mathfrak{F}\Phi^*$-hypercentral in G. This contradiction completes the proof. □

Proof of Theorem 1.22. (I) We first prove that E is p-nilpotent. Suppose that this is false and consider a counterexample (G, E) for which $|G| + |E|$ is minimal. Then:

(1) $O_{p'}(G) = 1$.

Suppose that $D = O_{p'}(G) \neq 1$. Then the hypothesis holds for $(G/D, ED/D)$ by Lemma 3.2(2). The choice of (G, E) implies that $E/E \cap D \simeq ED/D$ is p-nilpotent, which implies the p-nilpotency of E, a contradiction. Hence we have (1).

(2) *If $O_p(E) \neq 1$, then E is p-soluble.*

Indeed, by Lemma 3.2(2), the hypothesis holds for $(G/O_p(E), E/O_p(E))$. Hence in this case when $O_p(E) \neq 1$, $E/O_p(E)$ is p-nilpotent by the choice of (G, E), which implies the p-solubility of E.

(3) *$N \leqslant O_p(E)$ for every minimal normal subgroup N of G contained in E.*

Suppose that this is false, that is, there exists a minimal normal subgroup N of G contained in E such that $N \nleqslant O_p(E)$. Then by (1), N is non-abelian and by Feit–Thompson theorem $p = 2$ divides $|N|$. Moreover, in view of (2), $O_p(E) = O_2(E) = 1$.

Now let M be an arbitrary maximal subgroup of P. Let $(M, S, T) \in \Gamma_M$. Suppose that $S \neq 1$. By hypothesis, S is a Sylow 2-subgroup of some subnormal subgroup W of G. Let L be a minimal subnormal subgroup of G contained in W. If L is a $2'$-group, then $L \leqslant O_{2'}(G)$ by Lemma 3.4(3). But this contradicts (1). Hence 2 divides $|L|$ and $L_2 = S \cap L$ is a Sylow 2-subgroup of L since S is a Sylow 2-subgroup of W. Suppose that L is a 2-group. Then $L \leqslant O_2(G)$ by Lemma 3.4(3). It follows that $L \leqslant O_2(E) = 1$, a contradiction. Therefore L is a non-abelian simple group. Let $R = L^G$ be the normal closure of L in G. Then R is a minimal normal subgroup of G. Indeed, by Lemma 3.4(2), (5), for any $x, y \in G$ we have $\langle L^x, L^y \rangle = L^x \times L^y$. On the other hand, every normal subgroup D of R may be written in the form $L^x \times L^y \times \cdots \times L^z$ for some $x, y, \ldots, z \in G$ by [7, Chapter A, Theorem 4.13(b)]. Hence every normal subgroup of G contained in R is either 1 or R, and so $S \cap R \neq 1$. Since S satisfies the Φ^*-property in G, $|G : N_G(S \cap R)|$ is a power of 2. Therefore $R \leqslant (S \cap R)^G \leqslant O_2(G)$ by Lemma 3.1. Hence $O_2(E) \neq 1$. This contradiction shows that $S = 1$. Hence every maximal subgroup M of P has a supplement T in G such that $M \cap T \leqslant Z_{\mathfrak{F}}(T)$. We show that $V = T \cap E$ is 2-nilpotent. Let V_2 be a Sylow 2-subgroup of V containing $M \cap V$. Then $|V_2 : V \cap M| \leqslant |P : M| = 2$. Therefore for a Sylow 2-subgroup Q of $VZ_{\mathfrak{F}}(T)/Z_{\mathfrak{F}}(T)$ we have $|Q|$ divides 2. Hence $VZ_{\mathfrak{F}}(T)/Z_{\mathfrak{F}}(T) \simeq V/V \cap Z_{\mathfrak{F}}(T)$ is 2-nilpotent. It is well-known that the class of all 2-nilpotent groups is a hereditary saturated formation. Hence in view of Lemma 2.2 in [17], $V = T \cap E$ is 2-nilpotent. But $E = E \cap TM = M(T \cap E)$, so every maximal subgroup of P has a 2-nilpotent supplement T in E. It is clear that a Hall $2'$-subgroup of $T \cap E$ is also a Hall $2'$-subgroup of E. Hence by Corollary 3.7 and Lemma 3.8, E is 2-nilpotent. Hence R is a 2-group. This contradiction shows that (3) holds.

(4) *There is a maximal subgroup D of G such that $ND = G$ and $E = N \rtimes M$, where $M = D \cap E$ and $N = O_p(E) = C_E(N)$ is a minimal normal subgroup of G and M is p-nilpotent. In particular, E is p-soluble.*

In view of (3), $O_p(E) \neq 1$. Let N be a minimal normal subgroup of G contained in $O_p(E)$. Then the hypothesis holds for $(G/N, E/N)$ by Lemma 3.2(2). Therefore E/N is p-nilpotent by the choice of (G, E). It follows from (3) that N is the unique minimal normal subgroup of G contained in E. If $N \leqslant \Phi(G)$, then E is p-nilpotent by Corollary 1.6. Hence $N \nleqslant \Phi(G)$ and so $G = N \rtimes D$ for some maximal subgroup D of G. Since $O_p(G) \leqslant C_G(N)$ by Lemma 2.11, $O_p(G) \cap D$ is normal in G. Hence $O_p(G) \cap D \cap E$ is normal in G. Note that $E = N \rtimes (D \cap E)$, so

$$O_p(E) = O_p(G) \cap E = N\big(O_p(E) \cap D \cap E\big),$$

where $O_p(E) \cap D \cap E = O_p(G) \cap D \cap E$ is normal in G. Hence $O_p(E) \cap D \cap E = 1$, and so $N = O_p(E)$. Finally, since E is p-soluble and $O_{p'}(E) = 1$ by (3), we have $C_E(N) = N$ by [26, (9.3.1)].

(5) *If H/K is a chief factor of E below N, then $|H/K| > p$.*

By Proposition 4.13(c) in [7, Chapter A], $N = N_1 \times \cdots \times N_t$, where $N_1, \ldots, N_t$ are minimal normal subgroups of E, and from the proof of this proposition we see that $|N_i = |N_j|$ for all $i, j \in \{1, \ldots, t\}$. Hence for any chief factor H/K of E below N we have $|H/K| = |N_1|$ by Lemma 2.5. Suppose that $|H/K| = p$. Since $(p-1, |E|) = p$, $C_E(H/K) = E$. Hence $N \leqslant Z_\infty(E)$, which implies the p-nilpotency of E by (4). This contradiction shows that (5) holds.

(6) *If S is a non-identity subgroup of P satisfying the Φ^*-property in G, then $S \cap N \neq 1$.*

Indeed, let W be a subnormal subgroup of G such that S is a Sylow p-subgroup of W. If $S \cap N = 1$, then $W \cap N = 1$. Hence by (4) and Lemma 3.4(4), $W \leqslant C_E(N) = N$, a contradiction. Thus (6) holds.

(7) *$M = N_E(M_{p'})$, where $M_{p'}$ is the Hall p'-subgroup of M.*

Let $J = N_E(M_{p'})$. Suppose that $M < J$. Then $J = J \cap NM = M(J \cap N)$ and therefore $J \cap N \neq 1$. Since $E = NJ$, $J \cap N$ is normal in E and $E/C_E(J \cap N)$ is a p-group. Then in view of [7, Chapter A, Lemma 13.3], $J \cap N \leqslant Z_\infty(E)$. Hence for some minimal normal subgroup C of E contained in N we have $|C| = p$, which contradicts (5).

Final contradiction for (I). Let $M_p \leqslant D_p$, where M_p is a Sylow p-subgroup of M and D_p is a Sylow p-subgroup of D. Without loss of generality we may suppose that $M_p \leqslant P$. Then $NM_p = P$ and ND_p is a Sylow p-subgroup of G. Let $N_1 \leqslant N$ be a normal subgroup of ND_p such that $|N : N_1| = p$. Let $W = N_1 D_p$ and $V = N_1 M_p$. Then W is maximal in ND_p and V is maximal in P. By hypothesis, V is $\mathfrak{F}_{\Phi^*}$-supplemented in G. Let $(V, S, T) \in \Gamma_V$. First suppose that $S \neq 1$. Then $S \cap N \neq 1$ by (6). Since S satisfies the Φ^*-property in G, $|G : N_G(S \cap N)|$ is a power of p. Therefore $N \leqslant (S \cap N)^G \leqslant W$ by Lemma 3.1. Hence $ND_p = W$. This contradiction shows that $S = 1$. We show that in this case the subgroup $T_0 = T \cap E$ is p-nilpotent. First note that since $S = 1$ we have

$$V \cap T_0 = V \cap T \leqslant Z_{\mathfrak{F}}(T) \cap T_0 \leqslant Z_{\mathfrak{F}}(T_0)$$

by Lemma 2.2 in [17]. Hence, as in the proof of (3), one can show that $T_0 = T \cap E$ is p-nilpotent. Since $VT_0 = V(T \cap E) = VT \cap E = E$, a Hall p'-subgroup $T_{p'}$ of T_0 is a Hall p'-subgroup of E. By (4), E is p-soluble and so any two Hall p'-subgroups of E are conjugate in E. Then in view of Lemma 3.5 we may, without loss of generality, assume that $T_{p'} \leqslant M$, so $T_0 \leqslant M$ by (7). It follows that $E = VT_0 = VM$. But since $M_p \leqslant V$ and V is maximal in P, $VM \neq E$. The final contradiction shows that E is p-nilpotent.

(II) We now prove that $E/O_{p'}(E)$ is $\mathfrak{U}\Phi^*$-hypercentral in $G/O_{p'}(E)$.

In fact, by Lemma 3.2, the hypothesis holds for $(G/O_{p'}(E), E/O_{p'}(E))$. If $O_{p'}(E) \neq 1$, then $E/O_{p'}(E)$ is $\mathfrak{U}\Phi^*$-hypercentral in $G/O_{p'}(E)$ by the choice of (G, E). On the other hand, if $O_{p'}(E) = 1$, then E is a normal p-group of G and so E is $\mathfrak{F}\Phi^*$-hypercentral in G by Theorem 1.21(a). The theorem is proved. □

Proof of Theorem 1.21(b). Suppose that this assertion is false and consider a counterexample (G, E) for which $|G| + |E|$ is minimal. By Theorem 1.22 and [26, (10.1.9)], E is p-nilpotent, where p is the smallest prime dividing $|E|$. Let V be the normal Hall p'-subgroup of E. Then V is normal in G since it is characteristic in E. Moreover, by Lemma 3.2(2), the hypothesis holds for $(G/V, E/V)$. It is clear also that (G, V) is also satisfies the hypothesis. Hence in the case when $V \neq 1$, we have E/V is $\mathfrak{U}\Phi^*$-hypercentral in G/V and V is $\mathfrak{U}\Phi^*$-hypercentral in G by the choice of (G, E). This implies that E is $\mathfrak{U}\Phi^*$-hypercentral in G, a contradiction. This completes the proof. □

Proof of Theorem E. By Theorem 1.21(b), X is $\mathfrak{U}\Phi^*$-hypercentral in G. Thus in view of Theorem A and Corollary 1.5 we obtain that $G \in \mathfrak{F}$. □

4. Further applications

In the literature one can find many special cases of Theorems E, 1.21 and 1.22. Hear we list some of the most recent results.

Corollary 4.1. *(See [25].) Let P be a Sylow p-subgroup of a group G, where p is a prime divisor $|G|$ with $(p-1, |G|) = 1$. If all maximal subgroups of P are S-quasinormally embedded in G, then G is p-nilpotent.*

A subgroup H is said to be c^*-normal in G [38] if G has a normal subgroup T such that $HT = G$ and $H \cap T$ is S-quasinormally embedded in G. Clearly, every c^*-normal in G subgroup is $\mathfrak{F}_{\Phi^*}$-supplemented in G for any class $\mathfrak{F}$ of groups.

Corollary 4.2. *(See [38].) Let E be a normal subgroup of a group G with p-nilpotent quotient G/E, p a prime dividing $|E|$ with $(p-1, |G|) = 1$. Let P be a Sylow p-subgroup of E. If every maximal subgroup of P is c^*-normal in G, then G is p-nilpotent.*

A subgroup H of G is said to be weakly S-permutable (S-supplemented) in G [34] if there are a subgroup (a subnormal subgroup, respectively) T of G and an S-quasinormal subgroup S of G contained in H such that $HT = G$ and $H \cap T \leqslant S$.

Corollary 4.3. *(See [23].) Let P be a Sylow p-subgroup of a group G, where p is the smallest prime dividing $|G|$. If all maximal subgroups of P are weakly S-permutable in G, then G is p-nilpotent.*

Recall that a subgroup H of a group G said to be c-normal in G [37] if G has a normal subgroup T such that $HT = G$ and $H \cap T \leqslant H_G$.

Corollary 4.4. *(See [22].) Let $\mathfrak{F}$ be a saturated formation containing all supersoluble groups. If there exists a soluble normal subgroup E of G such that $G/E \in \mathfrak{F}$ and every maximal subgroup of any Sylow subgroup of E is either c-normal or S-quasinormally embedded in G, then $G \in \mathfrak{F}$.*

Corollary 4.5. *(See [22].) Let $\mathfrak{F}$ be a saturated formation containing all supersoluble groups. If there exists a soluble normal subgroup E of G such that $G/E \in \mathfrak{F}$ and every maximal subgroup of any Sylow subgroup of $F^*(E)$ is either c-normal or S-quasinormally embedded in G, then $G \in \mathfrak{F}$.*

Corollary 4.6. *(See [38].) Let $\mathfrak{F}$ be a saturated formation containing all supersoluble groups. If there exists a normal subgroup E of G such that $G/E \in \mathfrak{F}$ and every maximal subgroup of any Sylow subgroup of $F^*(E)$ is c^*-normal in G, then $G \in \mathfrak{F}$.*

Corollary 4.7. *(See [39].) Let $\mathfrak{F}$ be a saturated formation containing all supersoluble groups. If there exists a normal subgroup E of G such that $G/E \in \mathfrak{F}$ and every maximal subgroup of any Sylow subgroup of $F^*(E)$ is c-normal in G, then $G \in \mathfrak{F}$.*

Corollary 4.8. *(See [38].) Let $\mathfrak{F}$ be a saturated formation containing all supersoluble groups. If there exists a normal subgroup E of G such that $G/E \in \mathfrak{F}$ and every maximal subgroup of any Sylow subgroup of E is c^*-normal in G, then $G \in \mathfrak{F}$.*

Corollary 4.9. *(See [24].) Let $\mathfrak{F}$ be a saturated formation containing all supersoluble groups. If there exists a normal subgroup E of G such that $G/E \in \mathfrak{F}$ and every maximal subgroup of any Sylow subgroup of $F^*(E)$ is S-quasinormally embedded in G, then $G \in \mathfrak{F}$.*

Corollary 4.10. *(See [15].) A group G is supersoluble if and only if every maximal subgroup of every Sylow subgroup of G has a supersoluble supplement in G.*

Corollary 4.11. *(See [18].) A group G is supersoluble if and only if every maximal subgroup V of every Sylow subgroup of G either is normal or has a supplement T in G such that $V \cap T \leqslant Z_{\mathfrak{U}}(T)$.*

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